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Advanced Optimization:
Techniques and Industrial Applications

Curriculum Development
of Master's Degree Program in
Industrial Engineering for Thailand Sustainable Smart Industry

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Session 1.2:
Linear Programming

Universidade do Minho

Curriculum Development
of Master's Degree Program in
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Introduction

LP problem is an optimization problem in which

- The objective function is a linear function
- Each constraint is a linear equation or linear inequality

Example 1: Product Mix

Produce *n* products from *m* types of material.

Available on-hand inventory of material i $(i = 1,2,\ldots,m)$: b_i

Amount of material *i* used for one unit of product j $(j = 1,2,\ldots,n)$: a_{ij}

Profit of one unit of product *j*: c_j

Problem: determine production volumes of products so as to maximize total profit.

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Introduction

Denote x_j $(j = 1,2,..,n)$: production volume of product *j*.

Objective Function: Maximize $Z = \sum_{j=1}^{n} c_j x_j$

Constraints:

* Material constraints: $\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \forall i = 1,2,\ldots,m$

* Variable constraints: $x_j \geq 0 \quad \forall j = 1,2,\ldots,n$

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Introduction

Example 2: Diet Problem

Choose a diet from a set of *n* available foods in order to guarantee *m* nutritional requirements while minimizing cost

Daily required number of units of nutrient i $(i = 1,2, \ldots, m)$: b_i

Number of units of nutrient *i* in one unit of food *j*: a_{ij}

Cost per unit of food *j*: c_j

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Introduction

Denote x_j $(j = 1,2,..,n)$: number of units of food *j* in the diet

Objective Function: Minimize $Z = \sum_{j=1}^{n} c_j x_j$

Constraints:

* Nutrition constraints: $\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \forall i = 1,2,\dots,m$
* Variable constraints: $x_j \geq 0 \quad \forall j = 1,2,\dots,n$

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Introduction

Example 3: Transportation problem

There are *m* suppliers and *n* customers.

Supply capacity of supplier i $(i = 1,2,\ldots,m)$: s_i

Demand of customer j $(j = 1,2,\ldots,n)$: d_j

Variable cost of shipping one unit of goods from supplier *i* to customer *j* : c_{ij}

Determine the shipping plan so as to minimize total transportation cost.

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Introduction

Denote x_{ij} $(i = 1,2,..,m;\ j = 1,2,..,n)$: : number of units shipped from supplier *i* to customer *j*.

Objective Function: Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Constraints:

* Supply constraints: $\sum_{j=1}^{n} x_{ij} \leq s_i \quad \forall i = 1,2,\ldots,m$
* Demand constraints: $\sum_{i=1}^{m} x_{ij} \geq d_j \quad \forall j = 1,2,\ldots,n$
* Variable constraints: $x_{ij} \geq 0\ (i = 1,2,..,m;\ j = 1,2,..,n)$

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Background Concepts

Standard form of an LP problem

Minimize $\sum_{j=1}^{n} c_j x_j$

s.t. $\sum_{j=1}^{n} a_{ij} x_j = b_i \geq 0 \qquad \forall i = 1,2,\dots,m$

$x_j \geq 0 \qquad \forall j = 1,2,\dots,n$

(Note: some textbooks use "Maximization")

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Background Concepts

Matrix form

$$\begin{aligned} &\text{Minimize} && \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ &\text{s.t.} && \mathbf{A}\mathbf{x} = \mathbf{b} \geq \mathbf{0} \\ &&& \mathbf{x} \geq \mathbf{0} \end{aligned}$$

In which

- **c**, **x** are column vectors with dimension *n* ($\mathbf{c}, \mathbf{x} \in \mathbf{R}^{n}$)
- $\mathbf{c}^{\mathrm{T}}$ is the transpose of **c** - a row vector with dimension *n*
- **b** is a column vector with dimension *m* ($\mathbf{b} \in \mathbf{R}^{m}$)
- **A** is a *mxn* matrix ($\mathbf{A} \in \mathbf{R}^{mxn}$)

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Convert an LP to Standard Form

1. max $\sum_j c_j x_j \quad \Longrightarrow \quad$ min $\sum_j (-c_j) x_j$
2. $\sum_j a_{ij} x_j \leq b_i \quad \Longrightarrow \quad \sum_j a_{ij} x_j + y_i = b_i, y_i \geq 0$
   The added variable y_i: *slack* variable for the constraint
3. $\sum_j a_{ij} x_j \geq b_i \quad \Longrightarrow \quad \sum_j a_{ij} x_j - y_i = b_i, y_i \geq 0$
   The added variable y_i: *surplus* or *excess* variable
4. $\sum_j a_{ij} x_j = b_i, b_i < 0 \quad \Longrightarrow \quad \sum_j (-a_{ij}) x_j = -b_i$
5. If variable x_i can have either nonnegative or negative values then $x_i = x_i^{+} - x_i^{-}$ in which $x_i^{+}, x_i^{-} \geq 0$.

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Background Concepts

Example 4:

Consider the following LP:

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 1 \\ & 2x_1 - x_2 \ge 2 \\ & x_1 \ge 0 \end{aligned}$$

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Background Concepts

The standard form:

$$
\begin{aligned}
\min \quad & -3x_1 - 2x_2^+ + 2x_2^- \\
\text{s.t.} \quad & x_1 + x_2^+ - x_2^- + x_3 = 1 \\
& 2x_1 - x_2^+ + x_2^- - x_4 = 2 \\
& x_1, x_2^+, x_2^-, x_3, x_4 \geq 0
\end{aligned}
$$

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Background Concepts

Feasible Solution

Def.: Consider the LP model:

Minimize $\mathbf{c}^{\mathrm{T}}\mathbf{x}$

s.t. $\mathbf{Ax} = \mathbf{b} \geq \mathbf{0}$

$\mathbf{x} \geq \mathbf{0}$

Vector $\mathbf{x} \in \mathbf{R}^n$ which satisfies $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ is a *feasible solution* of the LP

- The set of all feasible solutions: *feasible region*

- If the feasible region does not exist, the LP problem is an *infeasible* problem

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Background Concepts

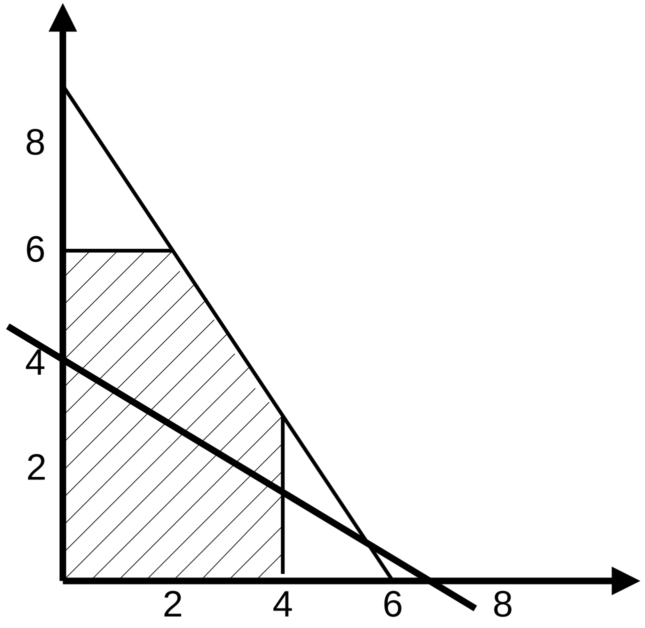
Example 5:

$$\max \ Z = 3x_1 + 5x_2$$

Constraints:

$$\begin{aligned} x_1 \quad & \leq 4 \\ 2x_2 \quad & \leq 12 \\ 3x_1 + 2x_2 \quad & \leq 18 \\ x_1, x_2 \geq 0 \end{aligned}$$

Feasible region of the model: *crossed* area in the figure



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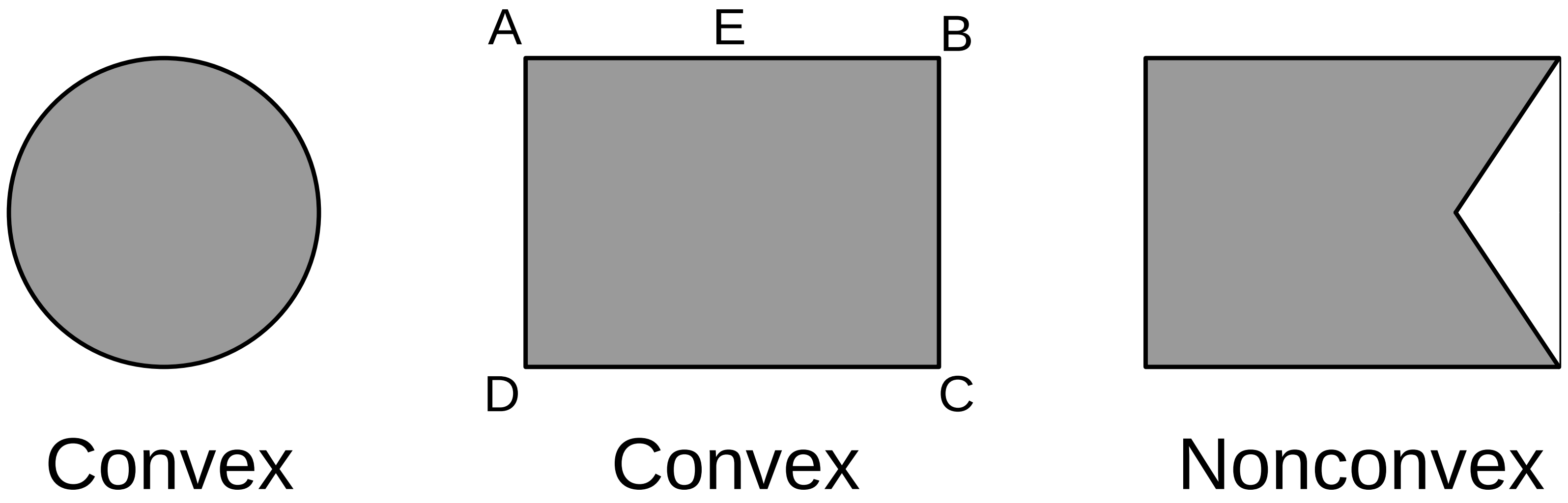
Background Concepts

Convex Set

Def. A set of points *S* is a convex set if the line segment connecting any pair of point in *S* is wholly contained in *S*



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Background Concepts

Extreme Point

Def. For any convex set, a point *P* in *S* is an extreme point if it does not lie on any line segment connecting two distinct points in the set

In the above figures, the extreme points are any point in the circumference of the circle (first figure) and A, B, C, D (second figure)

Note that

- The feasible region of an LP problem is a convex set
- The optimal solution of an LP problem is one of the extreme points of the feasible region

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Background Concepts

Basic and Nonbasic Variables

Consider an LP in standard form

$$
\begin{aligned}
\text{Min} \quad & Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
& \ldots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \\
& x_i \geq 0 \; (i = 1, 2, \ldots, n)
\end{aligned}
$$

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Background Concepts

Define:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

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Background Concepts

Suppose that $n \geq m$ and $rank(\mathbf{A}) = m$. If we set $(n-m)$ variables to zero and from $\mathbf{Ax} = \mathbf{b}$ we can find a unique solution for the remaining *m* variables then

- The $(n-m)$ variables that are set to zero are called ***nonbasic*** variables
- The *m* variables that have a unique solution are called ***basic*** variables and their solution is referred to as a ***basic solution***

<u>Note</u>: A basic solution is an extreme point of the feasible region

<u>In matrix form</u>: denote $\mathbf{A}_j$ - the *j*th column vector of **A**, we have

$$\mathbf{Ax} = \mathbf{b} \quad \Leftrightarrow \quad \sum_{j=1}^{n} \mathbf{A}_j x_j = \mathbf{b}$$

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Background Concepts

The subset of *m* column vectors associated with the *m* basic variables will form a ***basis*** **B**, and the corresponding basic solution can be found by solving

$$\mathbf{B}\mathbf{x}_{\mathbf{B}} = \mathbf{b} \quad \Rightarrow \; \mathbf{x}_{\mathbf{B}} = \mathbf{B}^{-1}\mathbf{b}$$

It is noted that

- The columns $\mathbf{A}_j$'s that form basis matrix **B** should be linearly independent.
- **B** is a nonsingular matrix, i.e., $\det(\mathbf{B}) \neq 0$

<u>Review</u>: The vectors $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ are linearly independent iff

$$\sum_{j=1}^{n} \alpha_j \mathbf{P}_j = 0 \qquad \Rightarrow \qquad \alpha_j = 0 \quad \forall j = 1,2,\ldots,n$$

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Simplex Method

Simplex tableau

Consider the following LP and its equivalent standard form

$$
\begin{aligned}
&\text{Min} \quad -2x_1 - x_2 \\
&\text{s.t} \\
&\quad x_1 + \frac{8}{3}x_2 \le 4 \\
&\quad x_1 + x_2 \le 2 \\
&\quad 2x_1 \le 3 \\
&\quad x_1, x_2 \ge 0
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
&\text{Min} \quad Z \\
&\text{s.t} \quad Z + 2x_1 + x_2 = 0 \\
&\quad x_1 + \frac{8}{3}x_2 + x_3 = 4 \\
&\quad x_1 + x_2 + x_4 = 2 \\
&\quad 2x_1 + x_5 = 3 \\
&\quad x_1, x_2 \ge 0
\end{aligned}
$$

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Simplex Method

The initial simplex tableau:

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	2	1	0	0	0	0
x_3	0	1	8/3	1	0	0	4
x_4	0	1	1	0	1	0	2
x_5	0	2	0	0	0	1	3

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Simplex Method

Notes:

- The variables associated with *unit column vectors* in the simplex tableau are current ***basic variables***. The others are ***nonbasic variables***.

- Initial feasible solution: $x_1 = x_2 = 0; x_3 = 4; x_4 = 2; x_5 = 3$
- Column vectors in matrix **A** associated with basic variables will form a ***basis matrix*** **B**. The basis **B** in a simplex tableau is an identity matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 8/3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{matrix} x_3 & x_4 & x_5 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Simplex Method

In general, a simplex tableau has the following form:

	Z	x_1	x_2	…	x_m	x_{B_1}	…	x_{B_j}	…	x_{B_r}	RHS
Z	1	y_{01}	y_{02}	…	y_{0m}	0	…	0	…	0	y_{00}
x_{B_1}	0	y_{11}	y_{12}	…	y_{1m}	1	…	0	…	0	y_{10}
…	…	…	…	…	…	…	…	…	…	…	…
x_{B_i}	0	y_{i1}	y_{i2}	…	y_{im}	0	…	1	…	0	y_{i0}
…	…	…	…	…	…	…	…	…	…	…	…
x_{B_r}	0	y_{r1}	y_{r2}	…	y_{rm}	0	…	0	…	1	y_{r0}

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Simplex Method

<u>Simplex Algorithm</u>

Assume that $y_{i0} \geq 0\ \forall i \in [1, r]$, at each iteration (called a *pivot*) of the simplex algorithm, the following steps will be performed

<u>Step 1</u>: select *j* such that $y_{oj} = max_{1 \leq k \leq m}\{y_{ok}\}$ - select the *<u>pivot column</u>*

- If $y_{oj} \leq 0$: stop. The current basic feasible solution is optimal.
- If $y_{oj} > 0$, go to step 2.

<u>Step 2</u>: for the value of *j* selected in step 1

- If $y_{ij} \leq 0\ \forall i$: stop; the LP problem is *<u>unbounded</u>*.
- Otherwise, select *i* such that – select the *<u>pivot row</u>*

$$\frac{y_{i0}}{y_{ij}} = Min\left\{\frac{y_{k0}}{y_{kj}}\middle| y_{kj} > 0\right\}$$ and go to step 3.

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Simplex Method

Step 3: Pivoting at y_{ij} (pivot term, pivot number) by use of elementary row operations as below and then go back to step 1,

Row *i* $\Rightarrow$ $\frac{1}{y_{ij}}$ ***(Row *i*)**

Row *k* $\Rightarrow$ **Row *k* -** $\frac{y_{kj}}{y_{ij}}$ ***(Row *i*)** $\forall k \neq i$

The purpose of step 3 is to replace the variable associated with row *i* (leaving variable) by the variable associated with column *j* (entering variable) in the current basis B in order to move from the current basic feasible solution to one of its adjacent basic feasible solutions .

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Simplex Method

Example 7:

1. $$\begin{aligned} &\text{Min} \quad -2x_1 - x_2 \\ &\text{s.t} \\ &x_1 + \frac{8}{3}x_2 \le 4 \\ &x_1 + x_2 \le 2 \\ &2x_1 \le 3 \\ &x_1, x_2 \ge 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\text{Min} \quad Z \\ &\text{s.t} \quad Z + 2x_1 + x_2 = 0 \\ &x_1 + \frac{8}{3}x_2 + x_3 = 4 \\ &x_1 + x_2 + x_4 = 2 \\ &2x_1 + x_5 = 3 \\ &x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

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Simplex Method

The initial simplex tableau:

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	2	1	0	0	0	0
x_3	0	1	8/3	1	0	0	4
x_4	0	1	1	0	1	0	2
x_5	0	2*	0	0	0	1	3

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Simplex Method

Step 1:

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	0	1	0	0	-1	-3
x_3	0	0	8/3	1	0	-1/2	5/2
x_4	0	0	1*	0	1	-1/2	1/2
x_1	0	1	0	0	0	1/2	3/2

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Simplex Method

Step 2:

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	0	0	0	-1	-1/2	-7/2
x_3	0	0	0	1	-8/3	5/6	7/6
x_2	0	0	1	0	1	-1/2	1/2
x_1	0	1	0	0	0	1/2	3/2

Optimal solution: $x_1^* = {}^3/_2,\ x_2^* = {}^1/_2,\ Z^* = -{}^7/_2$

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Simplex Method

Notes:

- A constraint is binding if its LHS and RHS are equal at the optimal solution. Otherwise, the constraint is nonbinding.

In the above example, constraint 1 is nonbinding while constraints 2&3 are binding

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Simplex Method

2. $$\begin{aligned} &\text{Min} \quad -2x_1 + x_2 \\ &\text{s.t} \\ &\qquad -x_1 + x_2 \le 2 \\ &\qquad x_1 - 3x_2 \le 3 \\ &\qquad x_1, x_2 \ge 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\text{Min} \quad Z \\ &\text{s.t} \quad Z + 2x_1 - x_2 = 0 \\ &\quad -x_1 + x_2 + x_3 = 2 \\ &\quad x_1 - 3x_2 + x_4 = 3 \\ &\quad x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

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Simplex Method

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	2	-1	0	0	0
x_3	0	-1	1	1	0	2
x_4	0	1*	-3	0	1	3

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Simplex Method

Step 1:

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	0	5	0	-2	-6
x_3	0	0	-2	1	1	5
x_1	0	1	-3	0	1	3

The problem is unbounded! Reason:

$$Z = -6 - 5x_2 + 2x_4$$
$$x_3 = 5 + 2x_2 - x_4 \qquad \Rightarrow \qquad Z \text{ can approach } -\infty$$
$$x_1 = 3 + 3x_2 - x_4$$

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Determine Initial Basic Solution

In some LP problems, after adding slack variables we still do not have an identity matrix to serve as an initial basis matrix (ill-behaved LPs). In these cases, an artificial starting solution should be determined so that the simplex algorithm can be applied.

There are two methods:

- The two-phase method
- The BigM method

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The Two-Phase Method

Phase 1: Introduce artificial variables in appropriate constraints of the LP problem so that an identity basis **B** can be formed. The objective function of phase 1 is to minimize the sum *W* of all artificial variables. Use simplex method for solution

There are three possibilities:

a. Case 1: If W^* exists but $W^* > 0 \Rightarrow$ The original LP is infeasible.

b. Case 2: If $W^* = 0$ and there is no artificial variable in the basis matrix. Remove all artificial variables from the current simplex tableau. Use this as the initial simplex tableau for the original LP, go to phase 2

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The Two-Phase Method

c. Case 3: If $W^* = 0$ and there are artificial variables in the basis matrix then:

- Remove the rows that have all elements associated with nonartificial variables to be zero.
- Pivot at some element (>0) in the current simplex tableau to take artificial variables out of the basis; remove all artificial variables from the current simplex tableau; use this as the initial simplex tableau for the original LP, and go to phase 2

Phase 2: Consider the original objective function and use simplex algorithm to find optimal solution.

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The Two-Phase Method

Example 8:

1.

$$\begin{aligned} &\text{Min} \quad -3x_1 + 4x_2 \\ &\text{s.t} \\ &\qquad x_1 + x_2 \le 4 \\ &\qquad 2x_1 + 3x_2 \ge 18 \\ &\qquad x_1, x_2 \ge 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\text{Min} \quad -3x_1 + 4x_2 \\ &\text{s.t} \\ &\qquad x_1 + x_2 + x_3 = 4 \\ &\qquad 2x_1 + 3x_2 - x_4 = 18 \\ &\qquad x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

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The Two-Phase Method

Phase 1: Min $W = x_5$

s.t. $x_1 + x_2 + x_3 = 4$

$2x_1 + 3x_2 - x_4 + x_5 = 18$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

	W	x_1	x_2	x_3	x_4	x_5	RHS
W	1	0	0	0	0	-1*	0
x_3	0	1	1	1	0	0	4
x_5	0	2	3	0	-1	1	18

This table is not a simplex tableau because the coefficients associated with the basic variables in row 0 are not zeros! So, Row operation should be conducted on row 0 to form the starting simplex tableau as follows

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The Two-Phase Method

Starting Simplex Tableau:

	W	x_1	x_2	x_3	x_4	x_5	RHS
W	1	2	3	0	-1	0	18
x_3	0	1	1*	1	0	0	4
x_5	0	2	3	0	-1	1	18

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The Two-Phase Method

Iteration 1: Pivot term: 1*; x_2 -entering variable; x_3 -leaving variable.

	W	x_1	x_2	x_3	x_4	x_5	RHS
W	1	-1	0	-3	-1	0	6
x_2	0	1	1	1	0	0	4
x_5	0	-1	0	-3	-1	1	6

Optimal solution has been found. However, $W^* = 6 \neq 0$. The original LP is *infeasible*.

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The Two-Phase Method

2. Min $4x_1 + x_2 + x_3$

s.t

$$2x_1 + x_2 + 2x_3 = 4$$

$$3x_1 + 3x_2 + x_3 = 3$$

$$x_1, x_2, x_3 \geq 0$$

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The Two-Phase Method

Phase 1: Min $W = x_4 + x_5$

s.t. $2x_1 + x_2 + 2x_3 + x_4 = 4$

$3x_1 + 3x_2 + x_3 + x_5 = 3$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

	W	x_1	x_2	x_3	x_4	x_5	RHS
W	1	0	0	0	-1	-1	0
x_4	0	2	1	2	1	0	4
x_5	0	3	3	1	0	1	3

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The Two-Phase Method

Starting Simplex Tableau:

	W	x_1	x_2	x_3	x_4	x_5	RHS
W	1	5	4	3	0	0	7
x_4	0	2	1	2	1	0	4
x_5	0	3*	3	1	0	1	3

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The Two-Phase Method

Iteration 1: Pivot term: 3*; x_1 -entering variable; x_5 -leaving variable.

	W	x_1	x_2	x_3	x_4	x_5	RHS
W	1	0	-1	4/3	0	-5/3	2
x_4	0	0	-1	4/3*	1	-2/3	2
x_1	0	1	1	1/3	0	1/3	1

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The Two-Phase Method

Iteration 2: Pivot term: 4/3*; x_3 -entering variable; x_4 -leaving variable.

	W	x_1	x_2	x_3	x_4	x_5	RHS
W	1	0	0	0	-1	-1	0
x_3	0	0	-3/4	1	3/4	-1/2	3/2
x_1	0	1	5/4	0	-1/4	1/2	1/2

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The Two-Phase Method

Phase 2:

	Z	x_1	x_2	x_3	RHS
Z	1	-4	-1	-1	0
x_3	0	0	-3/4	1	3/2
x_1	0	1	5/4	0	½

⇒ Starting simplex tableau of phase 2:

	Z	x_1	x_2	x_3	RHS
Z	1	0	13/4	0	7/2
x_3	0	0	-3/4	1	3/2
x_1	0	1	5/4*	0	1/2

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The Two-Phase Method

Iteration 1: Pivot term: 5/4*; x_2 -entering variable; x_1 -leaving variable.

	Z	x_1	x_2	x_3	RHS
Z	1	-13/5	0	0	11/5
x_3	0	3/5	0	1	9/5
x_2	0	4/5	1	0	2/5

Optimal solution has been founded!

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The Two-Phase Method

3. Min $-x_1 + 2x_2 - 3x_3$

s.t

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ -x_1 + x_2 + 2x_3 &= 4 \\ 2x_2 + 3x_3 &= 10 \\ x_3 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

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The Two-Phase Method

Phase 1: Min $W = x_5 + x_6 + x_7$

s.t.

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_5 &= 6 \\
-x_1 + x_2 + 2x_3 + x_6 &= 4 \\
2x_2 + 3x_3 + x_7 &= 10 \\
x_3 + x_4 &= 2 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0
\end{aligned}
$$

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The Two-Phase Method

Starting simplex tableau:

	W	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
W	1	0	4	6	0	0	0	0	20
x_5	0	1	1	1	0	1	0	0	6
x_6	0	-1	1	2*	0	0	1	0	4
x_7	0	0	2	3	0	0	0	1	10
x_4	0	0	0	1*	1	0	0	0	2

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The Two-Phase Method

Iteration 1: Pivot term: 1* (or 2*)

	W	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
W	1	0	4	0	-6	0	0	0	8
x_5	0	1	1	0	-1	1	0	0	4
x_6	0	-1	1*	0	-2	0	1	0	0
x_7	0	0	2	0	-3	0	0	1	4
x_3	0	0	0	1	1	0	0	0	2

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The Two-Phase Method

Iteration 2: Pivot term: 1*

	W	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
W	1	4	0	0	2	0	-4	0	8
x_5	0	2*	0	0	1	1	-1	0	4
x_2	0	-1	1	0	-2	0	1	0	0
x_7	0	2	0	0	1	0	-2	1	4
x_3	0	0	0	1	1	0	0	0	2

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The Two-Phase Method

Iteration 3: Pivot term: 2*

	W	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
W	1	0	0	0	0	-2	-2	0	0
x_1	0	1	0	0	1/2	1/2	-1/2	0	2
x_2	0	0	1	0	-3/2	1/2	1/2	0	2
x_7	0	0	0	0	0	-1	-1	1	0
x_3	0	0	0	1	1	0	0	0	2

In the final simplex tableau of phase 1 of this problem, the row associated with x_7 is redundant and should be removed

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The Two-Phase Method

Phase 2: The starting simplex tableau is developed as follows:

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	1	-2	3	0	0
x_1	0	1	0	0	1/2	2
x_2	0	0	1	0	-3/2	2
x_3	0	0	0	1	1	2

$\Rightarrow$

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	0	0	0	-13/2	-4
x_1	0	1	0	0	1/2	2
x_2	0	0	1	0	-3/2	2
x_3	0	0	0	1	1	2

Optimal solution found!

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The BigM Method

Procedure

- Introduce artificial variables into constraints so as an initial basic solution can be defined.
- For each artificial variable y_i, add an amount My_i (*M*: a very large positive value) to the objective function (the case of minimization problem)
- Use simplex method for solution.

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The BigM Method

Example 9:

1.

$$\begin{aligned}&\text{Min} \quad -3x_1 + 4x_2\\ &\text{s.t}\\ &\quad x_1 + x_2 \le 4\\ &\quad 2x_1 + 3x_2 \ge 18\\ &\quad x_1, x_2 \ge 0\end{aligned} \quad \Leftrightarrow \quad \begin{aligned}&\text{Min} \quad -3x_1 + 4x_2\\ &\text{s.t}\\ &\quad x_1 + x_2 + x_3 = 4\\ &\quad 2x_1 + 3x_2 - x_4 = 18\\ &\quad x_1, x_2, x_3, x_4 \ge 0\end{aligned}$$

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The BigM Method

$$\text{Min} \quad -3x_1 + 4x_2 + Mx_5$$

s.t

$$\Leftrightarrow \quad \begin{aligned} x_1 + x_2 + x_3 \qquad\qquad &= 4 \\ 2x_1 + 3x_2 \qquad - x_4 + x_5 &= 18 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	3	-4	0	0	-M	0
x_3	0	1	1	1	0	0	4
x_5	0	2	3	0	-1	1	18

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The BigM Method

Starting simplex tableau:

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	3+2M	-4+3M	0	-M	0	18M
x_3	0	1	1*	1	0	0	4
x_5	0	2	3	0	-1	1	18

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The BigM Method

Note: Another type of simplex tableau for BigM method:

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	3 2	-4 3	0 0	0 -1	0 0	0 18
x_3	0	1	1*	1	0	0	4
x_5	0	2	3	0	-1	1	18

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The BigM Method

Iteration 1: Pivot term: 1*; x_2 -entering variable; x_3 -leaving variable.

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	7-M	0	4-3M	-M	0	16+6M
x_2	0	1	1	1	0	0	4
x_5	0	-1	0	-3	-1	1	6

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The BigM Method

Or

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	7 -1	0 0	4 -3	0 -1	0 0	16 6
x_2	0	1	1	1	0	0	4
x_5	0	-1	0	-3	-1	1	6

The simplex tableau is optimal. But the artificial variable $x_5 > 0$
$\Rightarrow$ The original LP is infeasible.

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The BigM Method

2 $\quad \text{Min} \quad 4x_1 + x_2 + x_3$

s.t

$$2x_1 + x_2 + 2x_3 = 4$$

$$3x_1 + 3x_2 + x_3 = 3$$

$$x_1, x_2, x_3 \geq 0$$

$\Leftrightarrow \quad \text{Min} \quad 4x_1 + x_2 + x_3 + Mx_4 + Mx_5$

s.t.

$$2x_1 + x_2 + 2x_3 + x_4 = 4$$

$$3x_1 + 3x_2 + x_3 + x_5 = 3$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

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The BigM Method

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	-4	-1	-1	-M	-M	0
x_4	0	2	1	2	1	0	4
x_5	0	3	3	1	0	1	3

Starting simplex tableau:

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	-4+5M	-1+4M	-1+3M	0	0	7M
x_4	0	2	1	2	1	0	4
x_5	0	3*	3	1	0	1	3

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The BigM Method

<u>Iteration 1</u>: Pivot term: 3*; x_1 -entering variable; x_5 -leaving variable.

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	0	3-M	1/3+4M/3	0	4/3-5M/3	4+2M
x_4	0	0	-1	4/3*	1	-2/3	2
x_1	0	1	1	1/3	0	1/3	1

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The BigM Method

Iteration 2: Pivot term: 4/3*; x_3 -entering variable; x_4 -leaving variable.

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	0	13/4	0	-1/4-M	3/2-M	7/2
x_3	0	0	-3/4	1	3/4	-1/2	3/2
x_1	0	1	5/4*	0	-1/4	1/2	1/2

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The BigM Method

Iteration 3: Pivot term: 5/4*; x_2 -entering variable; x_1 -leaving variable.

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	-13/5	0	0	2/5-M	1/5-M	11/5
x_3	0	3/5	0	1	9/10	-1/5	9/5
x_2	0	4/5	1	0	-1/5	2/5	2/5

Optimal Solution: $x_1^* = 0, x_2^* = {}^2/_5, x_3^* = {}^9/_5$, with $Z^* = 11/5$.

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The BigM Method

3. Min $-x_1 + 2x_2 - 3x_3$ $\Leftrightarrow$ Min $-x_1 + 2x_2 - 3x_3 + Mx_5 + Mx_6 + Mx_7$

s.t

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ -x_1 + x_2 + 2x_3 &= 4 \\ 2x_2 + 3x_3 &= 10 \\ x_3 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

s.t.

$$\begin{aligned} x_1 + x_2 + x_3 + x_5 &= 6 \\ -x_1 + x_2 + 2x_3 + x_6 &= 4 \\ 2x_2 + 3x_3 + x_7 &= 10 \\ x_3 + x_4 &= 2 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0 \end{aligned}$$

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The BigM Method

	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
Z	1	1	-2	3	0	-M	-M	-M	0
x_5	0	1	1	1	0	1	0	0	6
x_6	0	-1	1	2	0	0	1	0	4
x_7	0	0	2	3	0	0	0	1	10
x_4	0	0	0	1*	1	0	0	0	2

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The BigM Method

Starting simplex tableau:

	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
Z	1	1	-2+4M	3+6M	0	0	0	0	20M
x_5	0	1	1	1	0	1	0	0	6
x_6	0	-1	1	2*	0	0	1	0	4
x_7	0	0	2	3	0	0	0	1	10
x_4	0	0	0	1*	1	0	0	0	2

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The BigM Method

Iteration 1: Pivot term: 1* (or 2*)

	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
Z	1	1	-2+4M	0	-3-6M	0	0	0	-6+8M
x_5	0	1	1	0	-1	1	0	0	4
x_6	0	-1	1*	0	-2	0	1	0	0
x_7	0	0	2	0	-3	0	0	1	4
x_3	0	0	0	1	1	0	0	0	2

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The BigM Method

Iteration 2: Pivot term: 1*

	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
Z	1	-1+4M	0	0	-7+2M	0	2-4M	0	-6+8M
x_5	0	2*	0	0	1	1	-1	0	4
x_2	0	-1	1	0	-2	0	1	0	0
x_7	0	2	0	0	1	0	-2	1	4
x_3	0	0	0	1	1	0	0	0	2

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The BigM Method

Iteration 3: Pivot term: 2*

	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
Z	1	0	0	0	-13/2	1/2-2M	3/2-2M	0	-4
x_1	0	1	0	0	1/2	1/2	-1/2	0	2
x_2	0	0	1	0	-3/2	1/2	1/2	0	2
x_7	0	0	0	0	0	-1	-1	1	0
x_3	0	0	0	1	1	0	0	0	2

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The BigM Method

Optimal Solution: $x_1^* = 2, x_2^* = 2, x_3^* = 2,$ with $Z^* = -4$.

Note that the information $x_7 = 0$ in the final simplex tableau is redundant (This must be satisfied!)

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Special Case in Simplex Method Application

There are four special cases:

- Degeneracy
- Alternative optima
- Unbounded solutions
- Infeasible solutions

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Special Case in Simplex Method Application
Degeneracy

When the model has at least one redundant constraint, a tie may occur when checking feasibility condition (Step 2) in the application of simplex method.

The tie can be broken arbitrarily and this will lead to the fact that at least one basic variable will be zero in the next iteration. This new solution is said to be ***degenerate***

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Special Case in Simplex Method Application
Degeneracy

Example 10:

$$\begin{aligned} &\text{Min} \quad z = -3x_1 - 9x_2 \\ &\text{s.t} \\ &\qquad x_1 + 4x_2 \leq 8 \\ &\qquad x_1 + 2x_2 \leq 4 \\ &\qquad x_1, x_2 \geq 0 \end{aligned}$$

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Special Case in Simplex Method Application
Degeneracy

Starting simplex tableau:

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	3	9	0	0	0
x_3	0	1	4*	1	0	8
x_4	0	1	2*	0	1	4

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Special Case in Simplex Method Application
Degeneracy

If x_2 enters and x_3 leaves the basic solution, then

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	3/4	0	-9/4	0	-18
x_2	0	1/4	1	1/4	0	2
x_4	0	1/2	0	-1/2	1	0

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Special Case in Simplex Method Application
Degeneracy

Next x_1 enters and x_4 leaves the basic solution, then

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	0	0	-3/2	-3/2	-18
x_2	0	0	1	1/2	-1/2	2
x_1	0	1	0	-1	2	0

Optimal solution! (but the value of objective function does not change)

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Special Case in Simplex Method Application
Alternative Optima

When the objective function is parallel to a *binding* constraint, the objective function will assume the *same optimal value*, called alternative optima, at more than one solution point. In this situation, there is an *infinity* of such solutions

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Special Case in Simplex Method Application
Alternative Optima

Example 11:

$$\begin{aligned}
&\text{Min} \quad z = -2x_1 - 4x_2 \\
&\text{s.t} \\
&\qquad x_1 + 2x_2 \le 5 \\
&\qquad x_1 + x_2 \le 4 \\
&\qquad x_1, x_2 \ge 0
\end{aligned}$$

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Special Case in Simplex Method Application
Alternative Optima

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	2	4	0	0	0
x_3	0	1	2*	1	0	5
x_4	0	1	1	0	1	4

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	0	0	-2	0	10
x_2	0	1/2	1	1/2	0	5/2
x_4	0	1/2	0	-1/2	1	3/2

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Special Case in Simplex Method Application
Alternative Optima

The optimal solution is reached. However, it is noted that the coefficient associated with nonbasic variable x_1 on row 0 is zero. Hence, x_1 can enter the basic solution without changing the optimal value of *z*.

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	0	0	-2	0	10
x_2	0	0	1	1	-1	1
x_1	0	1	0	-1	2	3

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Special Case in Simplex Method Application
Alternative Optima

Actually, any points in the segment connecting the two points $(x_1^1, x_2^1) = (0, \frac{5}{2})$ and $(x_1^2, x_2^2) = (3,1)$ will give the same objective value.

Optimal solutions:

$$(x_1, x_2) = \left(\alpha * 0 + (1-\alpha) * 3, \alpha * {}^5\!/_2 + (1-\alpha) * 1\right)$$

$$= \left(3 - 3\alpha, 1 + {}^{3\alpha}\!/_2\right) \forall \alpha \in [0,1]$$

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Special Case in Simplex Method Application
Unbounded Solution

When an LP model is poorly constructed (lack of some necessary constraints), the objective value may increase (in case of maximization) or decrease (in case of minimization) indefinitely

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Special Case in Simplex Method Application
Unbounded Solution

Example 12:

$$\begin{aligned}
\text{Min} \quad & z = -2x_1 - x_2 \\
\text{s.t} \quad & \\
& x_1 - x_2 \le 10 \\
& 2x_1 \qquad \le 40 \\
& x_1, x_2 \ge 0
\end{aligned}$$

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Special Case in Simplex Method Application
Unbounded Solution

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	2	1	0	0	0
x_3	0	1*	-1	1	0	10
x_4	0	2	0	0	1	40

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	0	3	-2	0	-20
x_1	0	1	-1	1	0	10
x_4	0	0	2*	-2	1	20

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Special Case in Simplex Method Application
Unbounded Solution

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	0	0	1	-3/2	-50
x_1	0	1	0	0	1/2	20
x_2	0	0	1	-1	1/2	10

In the last simplex tableau: $Z = -50 - x_3 + {}^{3}/_{2}\, x_4$

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Special Case in Simplex Method Application
Unbounded Solution

- For z to be minimized, x_4 should be zero. However, x_3 can increase indefinitely. In this case, the solution space is unbounded in the direction of x_3 and so the objective value.

- It is noted that, from the initial simplex tableau, we can see that all the constraint coefficients of x_2 are negative or zero. Therefore, x_2 can be increased indefinitely without violating any of the constraints and this will result in an infinite increase in z. In this case, the solution space is unbounded in the direction of x_3 and so the objective value.

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Special Case in Simplex Method Application
Unbounded Solution

How to recognize unboundedness? If at any iteration:

- All constraint coefficients of any nonbasic variable are zero or negative (unbounded solution space)
- The corresponding objective coefficient of that variable is also positive (unbounded objective value)

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Special Case in Simplex Method Application
Infeasible Solution

This is the case of LP models with inconsistent constraints (incorrectly formulated LPs)

See Example 8.1 or 9.1 discussed before for an illustration.

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Revised Simplex Method

Weaknesses of (Primal) Simple Method:

1. The initial basic solution comes only from the slack variables. So, it we don't have enough slack variables, some artificial variables should be introduced and then two-phase or BigM method should be applied. The increase in total number of variables will require more computational effort.
2. It is not flexible. We cannot select an arbitrary combination of variables to serve as a basic solution at the beginning. This flexibility is sometimes quite important if we already knew a near-optimal solution.

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Revised Simplex Method
SIMPLEX TABLE IN MATRIX FORM

Consider the LP: Minimize $z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$

s.t. $\mathbf{A}\mathbf{x} = \mathbf{b} \geq \mathbf{0}$

$\mathbf{x} \geq \mathbf{0}$

The problem can be written equivalently as: $\begin{pmatrix} 1 & -\mathbf{c}^{\mathrm{T}} \\ 0 & \mathbf{A} \end{pmatrix} \begin{pmatrix} z \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix}$

Suppose **B** is a feasible basis of the system $\mathbf{A}\mathbf{x} = \mathbf{b} \geq \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$;

Denote $\mathbf{x}_{\mathbf{B}}$: basic vector - the corresponding set of basic variables

$\mathbf{c}_{\mathbf{B}}$: the associated objective vector

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Revised Simplex Method
SIMPLEX TABLE IN MATRIX FORM

The corresponding feasible basic solution as well as the associated objective value can be determined as:

$$\begin{pmatrix} z \\ \mathbf{x}_B \end{pmatrix} = \begin{pmatrix} 1 & -\mathbf{c}_B^{\mathrm{T}} \\ 0 & \mathbf{B} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{pmatrix}$$

The general simplex tableau can be derived based on the following equation:

$$\begin{pmatrix} 1 & \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\mathbf{c}^{\mathrm{T}} \\ 0 & \mathbf{A} \end{pmatrix} \begin{pmatrix} z \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix}$$

Or equivalently,

$$\begin{pmatrix} 1 & \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^{\mathrm{T}} \\ 0 & \mathbf{B}^{-1}\mathbf{A} \end{pmatrix} \begin{pmatrix} z \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{pmatrix}$$

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Revised Simplex Method
SIMPLEX TABLE IN MATRIX FORM

Simplex tableau in matrix form:

	z	$\mathbf{x}$	RHS
z	1	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}^T$	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}\mathbf{b}$

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Revised Simplex Method
SIMPLEX TABLE IN MATRIX FORM

In details, the simplex tableau column associated with variable x_j can be represented as follows:

	z	x_j	RHS
z	1	$\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}_j - c_j$	$\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}_j$	$\mathbf{B}^{-1}\mathbf{b}$

Note that if x_j is a basic variable then: $\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}_j - c_j = 0$

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Revised Simplex Method
SIMPLEX TABLE IN MATRIX FORM

Example 13:

$$\text{Min} \quad -x_1 - 4x_2 - 7x_3 - 5x_4$$

s.t.

$$2x_1 + x_2 + 2x_3 + 4x_4 = 10$$

$$3x_1 - x_2 - 2x_3 + 6x_4 = 5$$

$$x_1, x_2, x_3, x_4 \geq 0$$

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Revised Simplex Method
SIMPLEX TABLE IN MATRIX FORM

Consider the simplex tableau associated with the basis $\mathbf{B} = (\mathbf{A}_1, \mathbf{A}_2)$

We have: $$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \mathbf{c}_B = \begin{bmatrix} -1 \\ -4 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \Rightarrow \mathbf{B}^{-1} = \begin{bmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{bmatrix}$$

Hence,

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

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Revised Simplex Method

SIMPLEX TABLE IN MATRIX FORM

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 4 \\ 3 & -1 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^{\mathrm{T}} = \begin{bmatrix} -1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} - \begin{bmatrix} -1 & -4 & -7 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 & 3 \end{bmatrix}$$

$$z = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} -1 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = -19$$

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Revised Simplex Method

SIMPLEX TABLE IN MATRIX FORM

The corresponding simplex tableau (without introducing any artificial variable !):

	Z	x_1	x_2	x_3	x_4	RHS
Z	1	0	0	-1	3	-19
x_1	0	1	0	0	2	3
x_2	0	0	1	2	0	4

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Revised Simplex Method
OPTIMALITY CONDITION

Consider a general LP: Minimize $z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$

s.t. $\mathbf{Ax} = \mathbf{b} \geq \mathbf{0}$

$\mathbf{x} \geq \mathbf{0}$

Row 0 of any simplex iterative can be represented by the following equation:

$$z + \sum_{j=1}^{n} \left(z_j - c_j\right) x_j = \mathbf{c}_B^{\mathrm{T}} \mathbf{B}^{-1} \mathbf{b}$$

In which $z_j - c_j = \mathbf{c}_B^{\mathrm{T}} \mathbf{B}^{-1} \mathbf{A}_j - c_j$

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Revised Simplex Method
OPTIMALITY CONDITION

In the above equation, it is noted that

- If x_j is a basic variable then $(z_j - c_j) = 0$
- If x_j is a nonbasic variable, an increase in x_j above its current zero value will improve the value of z only if $(z_j - c_j) > 0$.

Hence, optimal solution is achieved when: $(z_j - c_j) \leq 0 \ \forall j = 1,2, \dots, n$

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Revised Simplex Method
OPTIMALITY CONDITION

- When optimal condition is still not satisfied, any nonbasic variable satisfying $\left(z_j - c_j\right) > 0$ can be selected as entering variable to improve the current solution.

- The *rule of thumb* used in simplex method is to select the one with the most positive value of $\left(z_j - c_j\right)$ (in case of minimization).

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Revised Simplex Method
FEASIBILITY CONDITION

Row *i* of any simplex iteration can be represented by:

$$x_i + \sum_{j \neq i,\, j=1}^{n} \left(\mathbf{B}^{-1}\mathbf{A}_j\right)_i x_j = \left(\mathbf{B}^{-1}\mathbf{b}\right)_i$$

in which $\left(\mathbf{B}^{-1}\mathbf{A}_j\right)_i$, $(\mathbf{B}^{-1}\mathbf{b})_i$ are the elements of $\mathbf{B}^{-1}\mathbf{A}_j$, $\mathbf{B}^{-1}\mathbf{b}$ associated with row *i*.

When an $\mathbf{A}_j$ is selected to enter the basis, its associated nonbasic variable x_j will increase above zero level. At the same time, all other nonbasic variables remain at zero level. Therefore,

$$x_i = \left(\mathbf{B}^{-1}\mathbf{b}\right)_i - \left(\mathbf{B}^{-1}\mathbf{A}_j\right)_i x_j$$

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Revised Simplex Method
FEASIBILITY CONDITION

If $\left(\mathbf{B}^{-1}\mathbf{A}_j\right)_i > 0$, the increase in x_j should satisfy the following condition to ensure that $x_i \geq 0$.

$$x_j \leq \frac{\left(\mathbf{B}^{-1}\mathbf{b}\right)_i}{\left(\mathbf{B}^{-1}\mathbf{A}_j\right)_i}$$

The maximum value of the entering variable is, hence, determined by:

$$x_j = \min_i \left\{ \frac{\left(\mathbf{B}^{-1}\mathbf{b}\right)_i}{\left(\mathbf{B}^{-1}\mathbf{A}_j\right)_i} \middle| \left(\mathbf{B}^{-1}\mathbf{A}_j\right)_i > 0 \right\}$$

The basic variable associated with the minimum ratio will then leave the basic solution

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Revised Simplex Method

The revised simplex method is exactly the same as the tableau simplex method. The main difference is that it is based on matrix algebra while the tableau simplex method employs elementary row operations

Procedure:

1. Construct a starting basic feasible solution and its associated basis **B**
2. Compute the inverse $\mathbf{B}^{-1}$ by using an appropriate inversion method

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Revised Simplex Method

3. For each nonbasic variable x_j , compute

$$z_j - c_j = \mathbf{c}_B^{\mathrm{T}} \mathbf{B}^{-1} \mathbf{A}_j - c_j$$

If $(z_j - c_j) \leq 0$ for all nonbasic variables, stop; the optimal solution is given by

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}; \; z = \mathbf{c}_B^{\mathrm{T}} \mathbf{B}^{-1}\mathbf{b} = \mathbf{c}_B^{\mathrm{T}} \mathbf{x}_B$$

Else, select the entering variable x_j as the nonbasic variable with the most positive $(z_j - c_j)$

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Revised Simplex Method

4. Compute $\mathbf{B}^{-1}\mathbf{A}_j$

If all elements of $\mathbf{B}^{-1}\mathbf{A}_j$ are negative or zero, stop; the problem is unbounded

Else, compute $\mathbf{B}^{-1}\mathbf{b}$ and use the feasibility condition to determine the leaving variable among the current basic variables.

5. Form a new basis by replacing the leaving vector by the entering vector in the current basis **B**. Start a new iteration

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Revised Simplex Method

Example 14:

$$
\begin{aligned}
&\text{Min} \quad -5x_1 - 4x_2 \\
&\text{s.t.} \\
&6x_1 + 4x_2 + x_3 = 24 \\
&x_1 + 2x_2 + x_4 = 6 \\
&-x_1 + x_2 + x_5 = 1 \\
&x_2 + x_6 = 2 \\
&x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

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Revised Simplex Method

<u>Iteration 0</u>:

$$\mathbf{x}_{B_0} = \begin{bmatrix} x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T \quad \mathbf{c}_{B_0}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}_0 = \begin{bmatrix} \mathbf{A}_3 & \mathbf{A}_4 & \mathbf{A}_5 & \mathbf{A}_6 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{B}_0^{-1} = \mathbf{I}$$

Thus:

$$\mathbf{x}_{B_0} = \mathbf{B}_0^{-1}\mathbf{b} = \begin{bmatrix} 24 & 6 & 1 & 2 \end{bmatrix}^{\mathrm{T}}$$

$$z = \mathbf{c}_{B_0}^{\mathrm{T}}\mathbf{x}_{B_0} = 0$$

Check for optimality:

$$\mathbf{c}_{B_0}^{\mathrm{T}}\mathbf{B}_0^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left(z_j - c_j\right)_{j=1,2} = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}_0^{-1}\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix} - \begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \end{bmatrix}$$

$\Rightarrow$ x_1 is the entering variable

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Revised Simplex Method

Check feasibility condition: $\mathbf{x}_{B_0} = \mathbf{B}_0^{-1}\mathbf{b} = \begin{bmatrix} 24 & 6 & 1 & 2 \end{bmatrix}^{\mathrm{T}}$

$$\mathbf{B}_0^{-1}\mathbf{A}_1 = \begin{bmatrix} 6 & 1 & -1 & 0 \end{bmatrix}^{\mathrm{T}}$$

Hence, $x_1 = \min\left\{\frac{24}{6}, \frac{6}{1}, -, -\right\} = \min\{4, 6, -, -\} = 4$

$\Rightarrow$ x_3 is the leaving variable

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Revised Simplex Method

Iteration 1: $\mathbf{x}_{B_1} = [x_1 \quad x_4 \quad x_5 \quad x_6]^T \qquad \mathbf{c}_{B_1}^{\mathrm{T}} = [5 \quad 0 \quad 0 \quad 0]$

$$\mathbf{B}_1 = [\mathbf{A}_1 \quad \mathbf{A}_4 \quad \mathbf{A}_5 \quad \mathbf{A}_6] = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{B}_1^{-1} = \begin{bmatrix} 1/6 & 0 & 0 & 0 \\ -1/6 & 1 & 0 & 0 \\ 1/6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus $\mathbf{x}_{B_1} = \mathbf{B}_1^{-1}\mathbf{b} = [4 \quad 2 \quad 5 \quad 2]^{\mathrm{T}}$

$$z = \mathbf{c}_{B_1}^{\mathrm{T}} \mathbf{x}_{B_1} = -20$$

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Revised Simplex Method

Check for optimality: $\mathbf{c}_{B_1}^{\mathrm{T}}\mathbf{B}_1^{-1} = \begin{bmatrix} -5/6 & 0 & 0 & 0 \end{bmatrix}$

$$\left(z_j - c_j\right)_{j=2,3} = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}_1^{-1}\begin{bmatrix}\mathbf{A}_2 & \mathbf{A}_3\end{bmatrix} - \begin{bmatrix}c_2 & c_3\end{bmatrix} = \begin{bmatrix} 2/3 & -5/6 \end{bmatrix}$$

$\Rightarrow$ x_2 is the entering variable

Check feasibility condition: $\mathbf{x}_{B_1} = \mathbf{B}_1^{-1}\mathbf{b} = \begin{bmatrix} 4 & 2 & 5 & 2 \end{bmatrix}^{\mathrm{T}}$

$$\mathbf{B}_1^{-1}\mathbf{A}_2 = \begin{bmatrix} \frac{2}{3} & \frac{4}{3} & \frac{5}{3} & 1 \end{bmatrix}^{\mathrm{T}}$$

Hence, $x_2 = \min\left\{6, \frac{3}{2}, 3, 2\right\} = \frac{3}{2}$

$\Rightarrow$ x_4 is the leaving variable

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Revised Simplex Method

Iteration 2: $\mathbf{x}_{B_2} = \begin{bmatrix} x_1 & x_2 & x_5 & x_6 \end{bmatrix}^T \quad \mathbf{c}_{B_2}^{\mathrm{T}} = \begin{bmatrix} 5 & 4 & 0 & 0 \end{bmatrix}$

$$\mathbf{B}_2 = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_5 & \mathbf{A}_6 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{B}_2^{-1} = \begin{bmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{bmatrix}$$

Thus $\mathbf{x}_{B_2} = \mathbf{B}_2^{-1}\mathbf{b} = \begin{bmatrix} 3 & 3/2 & 5/2 & 1/2 \end{bmatrix}^{\mathrm{T}}$

$$z = \mathbf{c}_{B_2}^{\mathrm{T}} \mathbf{x}_{B_2} = -21$$

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Revised Simplex Method

Check for optimality: $\mathbf{c}_{B_2}^{\mathrm{T}}\mathbf{B}_2^{-1} = \begin{bmatrix} 3/4 & 1/2 & 0 & 0 \end{bmatrix}$

$$\left(z_j - c_j\right)_{j=3,4} = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}_2^{-1}\left[\mathbf{A}_3 \quad \mathbf{A}_3\right] - \left[c_3 \quad c_4\right] = \left[-3/4 \quad -1/2\right]$$

$\Rightarrow$ $\mathbf{x}_{B_2}$ is optimal. Optimal solution: $x_1 = 2, x_2 = 1.5, z = -21$

Note: Methods of determining the inverse of matrix: Adjoint Matrix Method, Gauss-Jordan Method, Use of Product Form of the Inverse (see textbook of Taha), LU Decomposition.

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The Dual of a Linear Program

Example 15: Consider the diet problem which is considered by a dieter

Choose a diet from a set of *n* available foods in order to guarantee *m* nutritional requirements while minimizing cost

- Daily required number of units of nutrient *i*: $b_i \ (i = 1,2,\dots,m)$
- Number of units of nutrient *i* in one unit of food *j* $(j = 1,2,\dots,n)$: a_{ij}
- Cost per unit of food *j* : $c_j \ (j = 1,2,\dots,n)$
- Number of units of food *j* in the diet: $x_j \ (j = 1,2,\dots,n)$

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The Dual of a Linear Program

The Primal Problem:

$$\textbf{(P)}: \quad \text{Min } Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad \forall i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad \forall j = 1, 2, \ldots, n$$

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The Dual of a Linear Program

Consider a druggist who sells *m* types of pill in which pill *i* contains one unit of nutrient *i* $(i = 1,2,\ldots,m)$. In order to convince the dieter to use his pills to supply the daily nutrient requirement, instead of using various foods, the prices of the pills $u_1, u_2, \ldots, u_m$ should be attractive in such a way that *the cost of a combination of m pills that provide exactly the same amount of nutrients as a unit of food j is less expensive than the cost of a unit of food j*.

If the dieter concerns about the minimum requirement of *m* nutrients, he will buy exactly b_i units of pill *i*.

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The Dual of a Linear Program

The problem of the druggist is to maximize his sales. This problem can be formulated as an LP problem as follows:

$$\textbf{(D):}\quad \text{Max } Z' = \sum_{i=1}^{m} b_i u_i$$

$$\text{s.t.}\quad \sum_{i=1}^{m} a_{ij} u_i \le c_j \qquad \forall j = 1, 2, \ldots, n$$

$$u_i \ge 0 \qquad \forall i = 1, 2, \ldots, m$$

(**P**) is the primal problem and (**D**) is the dual problem of (**P**).

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The Dual of a Linear Program

Example 16: Consider the Product Mix Problem

Company A want to produces *n* products from *m* types of material. The problem is to determine production volumes of products so as to maximize total profit

- Available on-hand inventory of material *i* : $b_i \ (i = 1,2, \dots, m)$
- Amount of material used for one unit of product *j* $(j = 1,2, \dots, n)$: a_{ij}
- Profit of one unit of product *j* : $c_j \ (j = 1,2, \dots, n)$
- Production volume of product j: $x_j \ (j = 1,2, \dots, n)$

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The Primal Problem:

$$(\mathbf{P}):\quad \text{Max } Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.}\quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad \forall i = 1,2,\ldots,m$$

$$x_j \ge 0 \qquad \forall j = 1,2,\ldots,n$$

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The Dual of a Linear Program

Suppose that company B wants to purchase all company A' resources. This request will be attractive to company A if

- The offered unit price of material *i* from company B is higher than the unit purchase price of material *i* an amount, says, u_i for each *i*.
- The profit comes from selling the raw materials needed to produce one unit of product *j* to company B should be higher than the profit gained from producing one unit of product *j*.

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The Dual of a Linear Program

The problem of company B is then to minimize the additional cost of purchasing while satisfying the above constraints:

$$(\mathbf{D}): \quad \text{Min } Z' = \sum_{i=1}^{m} b_i u_i$$

$$\text{s.t.} \quad \sum_{i=1}^{m} a_{ij} u_i \geq c_j \qquad \forall j = 1, 2, \ldots, n$$

$$u_i \geq 0 \qquad \forall i = 1, 2, \ldots, m$$

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The Dual of a Linear Program

In matrix form:

$$(\mathbf{P}):\quad \text{Min}\ \ \mathbf{c}^{\mathrm{T}}\mathbf{x} \qquad\qquad \Rightarrow \qquad\qquad (\mathbf{D}):\quad \text{Max}\ \ \mathbf{b}^{\mathrm{T}}\mathbf{u}$$

$$\text{s.t.}\quad \mathbf{A}\mathbf{x} \geq \mathbf{b} \qquad\qquad\qquad \text{s.t.}\quad \mathbf{A}^{T}\mathbf{u} \leq \mathbf{c}$$

$$\mathbf{x} \geq \mathbf{0} \qquad\qquad\qquad\qquad \mathbf{u} \geq \mathbf{0}$$

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The Dual of a Linear Program

Remarks:

1. The dual of the dual problem is the primal problem itself

Proof:

(D) is equivalent to: $-\text{Min } (-\mathbf{b})^T\mathbf{u}$

s.t. $(-\mathbf{A})^T\mathbf{u} \geq -\mathbf{c}$; $\mathbf{u} \geq \mathbf{0}$

and the dual is: $-\text{Max } (-\mathbf{c})^T\mathbf{x}$

s.t. $-\mathbf{Ax} \leq -\mathbf{b}$; $\mathbf{x} \geq \mathbf{0}$

which is equivalent to (P).

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The Dual of a Linear Program

2. The dual of the standard form LP:

$$(\mathbf{P}): \quad \text{Min} \ \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \ \mathbf{Ax} = \mathbf{b}, \ \mathbf{x} \ge \mathbf{0} \qquad \Rightarrow \qquad (\mathbf{D}): \quad \text{Max} \ \mathbf{b}^T\mathbf{u} \quad \text{s.t.} \ \mathbf{A}^T\mathbf{u} \le \mathbf{c}, \ \mathbf{u} \text{ unrestricted}$$

Proof:

$$(\mathbf{P}) \Leftrightarrow \begin{cases} \text{Min} \ \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \ \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \end{bmatrix} \mathbf{x} \ge \begin{bmatrix} \mathbf{b} \\ \mathbf{-b} \end{bmatrix} \\ \mathbf{x} \ge \mathbf{0} \end{cases}$$

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The Dual of a Linear Program

Hence,

$$(\mathbf{D}) \quad \Leftrightarrow \begin{cases} \text{Max} \ \ \mathbf{b}^T\mathbf{v} + (-\mathbf{b})^T \mathbf{w} \\ \text{s.t.} \ \ \begin{bmatrix} \mathbf{A}^T & -\mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} \le \mathbf{c} \\ \quad \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}^T \ge \mathbf{0} \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{Max} \ \ \mathbf{b}^T(\mathbf{v} - \mathbf{w}) \\ \text{s.t.} \ \ \mathbf{A}^T(\mathbf{v} - \mathbf{w}) \le \mathbf{c} \\ \quad \mathbf{v}, \mathbf{w} \ge \mathbf{0} \end{cases} \quad \Leftrightarrow \begin{cases} \text{Max} \quad \mathbf{b}^T\mathbf{u} \\ \text{s.t.} \quad \mathbf{A}^T\mathbf{u} \le \mathbf{c} \\ \quad \mathbf{u} \ \text{unrestricted} \end{cases}$$

(in which $\mathbf{u} = \mathbf{v} - \mathbf{w}$)

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The Dual of a Linear Program

3. In general, the conversion between the Primal and Dual can be summarized as follows:

Primal/Dual (Min)	Dual/Primal (Max)
$\geq b_i$	$u_i \geq 0$
$\leq b_i$	$u_i \leq 0$
$= b_i$	u_i unrestricted
$x_i \geq 0$	$\leq c_i$
$x_i \leq 0$	$\geq c_i$
x_i unrestricted	$= c_i$

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The Dual of a Linear Program

Example 17:

Primal Problem

$$
\begin{aligned}
&\textbf{(P)} \quad \text{Min} \quad -2x_1 + x_2 - 3x_3 + 2x_4 \\
&\text{s.t.} \\
&\quad x_1 + x_2 - 3x_3 - x_4 = 2 \\
&\quad -2x_1 - x_2 + 5x_3 - 4x_4 \geq 3 \\
&\quad 5x_1 + 3x_2 + x_3 - 2x_4 \leq -2 \\
&\quad x_1, x_2 \geq 0, x_3 \leq 0
\end{aligned}
$$

Dual Problem

$$
\begin{aligned}
&\textbf{(D)} \quad \text{Max} \quad 2u_1 + 3u_2 - 2u_3 \\
&\text{s.t.} \\
&\quad u_1 - 2u_2 + 5u_3 \leq -2 \\
&\quad u_1 - u_2 + 3u_3 \leq 1 \\
&\quad -3u_1 + 5u_2 + u_3 \geq -3 \\
&\quad -u_1 - 4u_2 - 2u_3 = 2 \\
&\quad u_2 \geq 0, u_3 \leq 0
\end{aligned}
$$

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Dual Theorem

Consider the primal and dual problems in the form (standard!):

$$(\mathbf{P}): \quad \text{Min } \mathbf{c}^{T}\mathbf{x} \qquad\qquad \Rightarrow \qquad\qquad (\mathbf{D}): \quad \text{Max } \mathbf{b}^{T}\mathbf{u}$$

$$\text{s.t.} \quad \mathbf{A}\mathbf{x} \geq \mathbf{b} \qquad\qquad\qquad\qquad \text{s.t.} \quad \mathbf{A}^{T}\mathbf{u} \leq \mathbf{c}$$

$$\mathbf{x} \geq \mathbf{0} \qquad\qquad\qquad\qquad\qquad \mathbf{u} \geq \mathbf{0}$$

We says that

$\bar{\mathbf{x}}$ is a primal feasible solution if $\mathbf{A}\bar{\mathbf{x}} \geq \mathbf{b}$ and $\bar{\mathbf{x}} \geq \mathbf{0}$

$\bar{\mathbf{u}}$ is a dual feasible solution if $\mathbf{A}^{T}\bar{\mathbf{u}} \leq \mathbf{c}$ and $\bar{\mathbf{u}} \geq \mathbf{0}$

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Dual Theorem
Weak Duality Theorem

If $\bar{\mathbf{x}}$ is a primal feasible solution and $\bar{\mathbf{u}}$ is a dual feasible solution then

$$\mathbf{c}^T\bar{\mathbf{x}} \geq \mathbf{b}^T\bar{\mathbf{u}}$$

Proof:

$$\begin{aligned}
\mathbf{c}^T\bar{\mathbf{x}} &\geq (\mathbf{A}^T\bar{\mathbf{u}})^T\bar{\mathbf{x}} && \text{since} \quad \mathbf{A}^T\bar{\mathbf{u}} \leq \mathbf{c}\ ,\bar{\mathbf{x}} \geq \mathbf{0} \\
&= \bar{\mathbf{u}}^T\mathbf{A}\bar{\mathbf{x}} \\
&\geq \bar{\mathbf{u}}^T\mathbf{b} && \text{since} \quad \mathbf{A}\bar{\mathbf{x}} \geq \mathbf{b}\ ,\bar{\mathbf{u}} \geq \mathbf{0} \\
&= (\bar{\mathbf{u}}^T\mathbf{b})^T = \mathbf{b}^T\bar{\mathbf{u}}
\end{aligned}$$

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Dual Theorem
Weak Duality Theorem

<u>*Corollaries*</u>:

- If $\bar{\mathbf{x}}$ is a primal feasible solution and $\bar{\mathbf{u}}$ is a dual feasible solution such that $\mathbf{c}^T\bar{\mathbf{x}} = \mathbf{b}^T\bar{\mathbf{u}}$ then $\bar{\mathbf{x}}$ is an optimal solution of (**P**) and $\bar{\mathbf{u}}$ is an optimal solution of (**D**) with the same optimal objective value: $\mathbf{c}^T\bar{\mathbf{x}} = \mathbf{b}^T\bar{\mathbf{u}}$

- If (**P**) (or (**D**)) is unbounded then (**D**) (or (**P**)) is infeasible

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Dual Theorem
Strong Duality Theorem

If both the primal and the dual problems are feasible, then both have optimal solutions $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ that satisfy $\mathbf{c}^T\bar{\mathbf{x}} = \mathbf{b}^T\bar{\mathbf{u}}$

Proof:

1. If (**P**) (or (**D**)) is feasible but does not have optimal solution then (**P**) (or (**D**)) is unbounded. Hence, (**D**) (or (**P**)) is infeasible! So, (**P**) (or (**D**)) should have optimal solution.

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Dual Theorem
Strong Duality Theorem

2. (P) is equivalent to: Min $\mathbf{c}^T\mathbf{x}$

s.t. $\mathbf{A}\mathbf{x}-\mathbf{y}=\mathbf{b}$ and $\mathbf{x},\mathbf{y}\geq\mathbf{0}$

Suppose $\bar{\mathbf{x}}$ is the optimal solution of (**P**) and the optimal simplex tableau is:

	z	$\mathbf{x}$	$\mathbf{y}$	RHS
z	1	$\bar{\mathbf{c}}_B^T\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}^T$	$-\bar{\mathbf{c}}_B^T\mathbf{B}^{-1}$	$\bar{\mathbf{c}}_B^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$-\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

In which $\bar{\mathbf{c}}=\begin{bmatrix}\mathbf{c}\\ \mathbf{c}_\mathbf{y}\end{bmatrix}=\begin{bmatrix}\mathbf{c}\\ \mathbf{0}\end{bmatrix}$

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Dual Theorem
Strong Duality Theorem

Note that the coefficient of the column associated with $\mathbf{y}$ is:

$$\bar{\mathbf{c}}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}_{\mathbf{y}} - \mathbf{c}_{\mathbf{y}}^{\mathrm{T}} = \bar{\mathbf{c}}_B^{\mathrm{T}}\mathbf{B}^{-1}\begin{bmatrix} -1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 \end{bmatrix} - \mathbf{0}^{\mathrm{T}} = -\bar{\mathbf{c}}_B^{\mathrm{T}}\mathbf{B}^{-1}$$

$$\mathbf{B}^{-1}\mathbf{A}_{\mathbf{y}} = -\mathbf{B}^{-1}$$

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Dual Theorem
Strong Duality Theorem

Let $\bar{\mathbf{u}} = \left(\bar{\mathbf{c}}_{\mathbf{B}}^T \mathbf{B}^{-1}\right)^T$, we have:

$$\bar{\mathbf{c}}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{A} - \mathbf{c}^T \leq \mathbf{0} \text{ and } -\bar{\mathbf{c}}_{\mathbf{B}}^T \mathbf{B}^{-1} \leq \mathbf{0} \text{ (due to optimality of (}\mathbf{P}\text{))}$$

$$\Longrightarrow \quad \mathbf{A}^T \bar{\mathbf{u}} \leq \mathbf{c} \text{ and } \bar{\mathbf{u}} \geq \mathbf{0}$$

$$\Longrightarrow \quad \bar{\mathbf{u}} \text{ is a feasible solution of (}\mathbf{D}\text{) that satisfies } \mathbf{b}^T \bar{\mathbf{u}} = \mathbf{c}^T \bar{\mathbf{x}}$$

From the weak duality theorem, it can be concluded that $\bar{\mathbf{u}}$ is an optimal solution of (**D**).

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Dual Theorem
Complementary Slackness Condition

If $\mathbf{x}$ and $\mathbf{u}$ are primal and dual feasible solutions then $\mathbf{x}$ and $\mathbf{u}$ are both optimal if and only if

$$\mathbf{u}^T(\mathbf{Ax}-\mathbf{b}) = 0 \text{ and } (\mathbf{A}^T\mathbf{u}-\mathbf{c})^T\mathbf{x} = \mathbf{0}$$

i.e.,

$$\sum_{j=1}^{n} a_{ij}x_j > b_i \Rightarrow u_i = 0$$

and

$$\sum_{i=1}^{m} a_{ij}u_i < c_j \Rightarrow x_j = 0$$

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Dual Theorem
Complementary Slackness Condition

Proof:

If $\mathbf{x}$ and $\mathbf{u}$ are both feasible, we have:

$$\mathbf{Ax} - \mathbf{b} \geq \mathbf{0} \quad \Longrightarrow \quad \mathbf{u}^T(\mathbf{Ax} - \mathbf{b}) \geq 0 \quad \Longrightarrow \quad \mathbf{u}^T\mathbf{Ax} \geq \mathbf{u}^T\mathbf{b}$$

$$\mathbf{A}^T\mathbf{u} - \mathbf{c} \leq \mathbf{0} \quad \Longrightarrow \quad (\mathbf{A}^T\mathbf{u} - \mathbf{c})^T\mathbf{x} \leq 0 \quad \Longrightarrow \quad \mathbf{u}^T\mathbf{Ax} \leq \mathbf{c}^T\mathbf{x}$$

1. If $\mathbf{x}$ and $\mathbf{u}$ are both optimal, we have: $\mathbf{u}^T\mathbf{b} = \mathbf{c}^T\mathbf{x}$, and hence, $\mathbf{u}^T(\mathbf{Ax} - \mathbf{b}) = 0$ and $(\mathbf{A}^T\mathbf{u} - \mathbf{c})^T\mathbf{x} = 0$
2. If $\mathbf{u}^T(\mathbf{Ax} - \mathbf{b}) = 0$ and $(\mathbf{A}^T\mathbf{u} - \mathbf{c})^T\mathbf{x} = 0$: $\mathbf{u}^T\mathbf{Ax} = \mathbf{u}^T\mathbf{b}$ and $\mathbf{u}^T\mathbf{Ax} = \mathbf{c}^T\mathbf{x}$. Therefore, $\mathbf{x}$ and $\mathbf{u}$ are both optimal solutions of (**P**) and (**D**).

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Relationship between Primal – Dual Solution

Consider an iterative of the simplex method applied on a standard LP with the current basis **B** and the associated simplex table:

	z	x	RHS
z	1	$\bar{\mathbf{c}}_B^T \mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T$	$\bar{\mathbf{c}}_B^T \mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B$	**0**	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}\mathbf{b}$

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Relationship between Primal – Dual Solution

- At each step of the simplex algorithm, we keep $\mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$, and thus the basic solution $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix} \geq \mathbf{0}$: always feasible.
- Let $\mathbf{u} = \left(\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\right)^T$ then $\mathbf{u}$ is *dual feasible* if and only if $\mathbf{A}^T\mathbf{u} \leq \mathbf{c}$, i.e., $\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T \leq \mathbf{0}$. This is the *optimality condition* of the primal problem.

The primal optimality condition is actually the dual feasibility condition

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Relationship between Primal – Dual Solution

Hence, at any intermediate stage of the simplex method, we have:

(i) A primal basic feasible solution $\mathbf{x} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix}$ and

(ii) A dual infeasible solution $\mathbf{u} = \left(\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1}\right)^T$

and Primal Obj. Value $= \mathbf{c}^T\mathbf{x} = \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1}\mathbf{b} = \mathbf{u}^T\mathbf{b} =$ Dual Obj. Value.

At the final simplex tableau, the dual solution become *dual feasible*.

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Relationship between Primal – Dual Solution

At optimal solutions $\mathbf{x}$ and $\mathbf{u}$, the optimal objective value is $z^* = \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{u} = \bar{\mathbf{c}}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{b}$. If b_i increases one unit, the optimal objective value will change and $\frac{dz^*}{db_i} = \left(\bar{\mathbf{c}}_{\mathbf{B}}^T\mathbf{B}^{-1}\right)_i = \bar{u}_i$. So, $\bar{u}_i$ is the rate of change of the optimal objective value z^* with respect to the change of right-hand side value b_i .

If constraint *i* is for a kind of resource,

$\bar{u}_i$ is called the shadow price of that resource

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Dual Simplex Method

In some LP problems, it is easy to find an initial simplex tableau which satisfies optimality conditions (or dual feasibility conditions) but does not satisfy feasibility conditions (or dual optimality conditions). For example,

$$\begin{array}{ll} \text{Minimize} & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

By introduce the surplus variables $\mathbf{y}$, an initial infeasible simplex tableau can be derived

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Dual Simplex Method

	z	$\mathbf{x}$	$\mathbf{y}$	RHS
z	1	$-\mathbf{c}^{T}$	$\mathbf{0}$	0
$\mathbf{x}_B$	$\mathbf{0}$	$-\mathbf{A}$	$\mathbf{I}$	$-\mathbf{b}$

The initial basic solution $\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ -\mathbf{b}\end{bmatrix}$ is infeasible but the optimality condition $[-\mathbf{c}^T \quad \mathbf{0}] \leq \mathbf{0}$ can be satisfied.

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Dual Simplex Method

In this case, if the simplex method introduced before is employed (primal simplex method), some artificial variables have to be introduced and it becomes more complicated to find the optimal solution (two-phase or Big M methods should be applied)

Starting with an optimal but infeasible simplex tableau, i.e., $y_{0i} \leq 0, \forall i \in [1, m]$, the dual simplex method can be applied to find optimal solution.

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Dual Simplex Method
PROCEDURE

Step 1: If $y_{j0} \geq 0, \forall i \in [1, r]$: stop, an optimal solution has been found. Otherwise, select i such that $y_{i0} = \min_{1 \leq k \leq r} \{y_{k0}\} < 0$.

Step 2: With the selected i, if $y_{ij} \geq 0, \forall j \in [1, m]$: stop, the primal problem is infeasible. Otherwise, select j such that

$$\frac{y_{0j}}{y_{ij}} = \min_{k} \left\{ \frac{y_{0k}}{y_{ik}} \middle| y_{ik} < 0 \right\}$$ and go to step 3.

Step 3: Pivot at y_{ij} and go back to step 1.

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Dual Simplex Method

Example 18:

$$\begin{aligned} \text{Min} \quad & 3x_1 + 4x_2 + 5x_3 \\ \text{s.t.} \quad & \\ & x_1 + 2x_2 + 3x_3 \geq 5 \\ & 2x_1 + 2x_2 + x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Introduce surplus variables:

$$\begin{aligned} \text{Min} \quad & 3x_1 + 4x_2 + 5x_3 \\ \text{s.t.} \quad & \\ & -x_1 - 2x_2 - 3x_3 + x_4 = 5 \\ & -2x_1 - 2x_2 - x_3 + x_5 = 6 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

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Dual Simplex Method

Initial infeasible simplex tableau:

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	-3	-4	-5	0	0	0
x_4	0	-1	-2	-3	1	0	-5
x_5	0	**-2***	-2	-1	0	1	-6

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Dual Simplex Method

Iteration 1:

- Min{-5,-6} = -6
- Min{-3/-2, -4/-2, -5/-1} = 3/2

⇒ Pivot term -2*, x_1 - entering variable; x_5 - leaving variable.

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	0	-1	-7/2	0	-3/2	9
x_4	0	0	**-1***	-5/2	1	-1/2	-2
x_1	0	1	1	1/2	0	-1/2	3

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Dual Simplex Method

Iteration 2:

- The only negative RHS value: -2
- Min{-1/-1, (-7/2)/(-5/2), (-3/2)/(-1/2)} = 1/1

⇒ Pivot term -1*, -entering variable; -leaving variable.

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	0	0	-1	-1	-1	11
x_2	0	0	1	5/2	-1	1/2	2
x_1	0	1	0	-2	1	-1	1

Optimal Solution:

$x_1^* = 1, x_2^* = 2, x_3^* = 0$

Objective Value = 11

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Dual Simplex Method

Note 1: A simplex table of the dual simplex algorithm for a given basis **B** has the form:

	Z	x_1	x_2	$\dots$	x_j	$\dots$	x_m	RHS
Z	1	y_{01}	y_{02}	$\dots$	y_{0j}	$\dots$	y_{0m}	y_{00}
x_{B_i}	0	y_{i1}	y_{i2}	$\dots$	y_{ij}	$\dots$	y_{im}	y_{i0}

$$z = y_{00} - \sum_{k \in D} y_{0k} x_k \qquad x_{\mathbf{B}_i} = y_{i0} - \sum_{k \in D} y_{ik} x_k$$

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Dual Simplex Method

If $y_{i0} < 0$ and $y_{ij} \geq 0, \forall j = 1,2,\dots,m$ then the primal problem is infeasible and hence, the dual problem is unbounded.

Proof: Since all $x_j \geq 0 \;\Rightarrow x_{\boldsymbol{B}_i} = y_{i0} - \sum_{k \in D} y_{ik} x_k < 0$: the primal problem is infeasible

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Dual Simplex Method

<u>Note 2</u>: In each iteration of the dual simplex method, primal optimality condition (dual feasibility condition) is always satisfied and the objective value does not decrease.

<u>Proof</u>: when pivoting at y_{ij}, the new objective value z' can be expressed as:

$$z' = y_{00} - y_{0j}\frac{y_{i0}}{y_{ij}} = y_{00} - y_{0j}x_j$$

Due to $y_{0j} < 0$ and $x_j \geq 0$: $z' = y_{00} - y_{0j}x_j \geq y_{00}$

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Dual Simplex Method

Furthermore, the new value of y_{0k}, denoted by y'_{ok} can be expressed as: $y'_{ok} = y_{0k} - \frac{y_{ik}}{y_{ij}} y_{0j}$. Hence,

If $y_{ik} > 0$: $\quad -\frac{y_{ik}}{y_{ij}} y_{0j} < 0 \quad \Rightarrow \quad y'_{ok} \leq y_{0k} \leq 0$

If $y_{ik} < 0$: $\quad y'_{ok} = y_{ik} \left(\frac{y_{0k}}{y_{ik}} - \frac{y_{0j}}{y_{ij}} \right) \leq 0$

due to $\frac{y_{0j}}{y_{ij}} = \min_k \left\{ \frac{y_{0k}}{y_{ik}} \middle| y_{ik} < 0 \right\}$

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Dual Simplex Method

Note 3: The dual simplex method is just simply the simplex method applied to the dual problem by using the primal simplex tableau.

Note 4: The optimal solution of the dual problem (if it exists) can be determined from the shadow prices of the primal problem.

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Dual Simplex Method

Example 19:

$$\textbf{(P):}\quad \text{Min} \quad -2x_1 - x_2$$

s.t

$$x_1 + \frac{8}{3}x_2 \le 4$$

$$x_1 + x_2 \le 2$$

$$2x_1 \le 3$$

$$x_1, x_2 \ge 0$$

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Dual Simplex Method

The optimal simplex tableau of this LP (see example 7):

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	0	0	0	-1	-1/2	-7/2
x_3	0	0	0	1	-8/3	5/6	7/6
x_2	0	0	1	0	1	-1/2	1/2
x_1	0	1	0	0	0	1/2	3/2

Optimal solution: $x_1^* = {}^3/_2,\ x_2^* = {}^1/_2,\ Z^* = -{}^7/_2$

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Dual Simplex Method

The dual problem:

$$\textbf{(D):}\quad \text{Max}\ \ 4u_1 + 2u_2 + 3u_3$$

s.t

$$u_1 + u_2 + 2u_3 \le -2$$

$$\frac{8}{3}u_1 + u_2 \qquad\qquad \le -1$$

$$u_1, u_2, u_3 \le 0$$

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Dual Simplex Method

Denote $U_1 = -u_1, U_2 = -u_2, U_3 = -u_3$, we have

$$
\begin{aligned}
&\text{Min} \quad 4U_1 + 2U_2 + 3U_3 \\
&\text{s.t} \\
&\qquad -U_1 - U_2 - 2U_3 \le -2 \\
&\qquad -\frac{8}{3}U_1 - U_2 \qquad\quad \le -1 \\
&\qquad U_1, U_2, U_3 \ge 0
\end{aligned}
$$

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Dual Simplex Method

Introduce slack variables:

$$
\begin{aligned}
&\text{Min} \quad 4U_1 + 2U_2 + 3U_3 \\
&\text{s.t} \\
&\quad -U_1 - U_2 - 2U_3 + U_4 = -2 \\
&\quad -\frac{8}{3}U_1 - U_2 + U_5 = -1 \\
&\quad U_1, U_2, U_3, U_4, U_5 \geq 0
\end{aligned}
$$

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Dual Simplex Method

Apply dual simplex method - the initial simplex tableau is

	Z	U_1	U_2	U_3	U_4	U_5	RHS
Z	1	-4	-2	-3	0	0	0
U_4	0	-1	-1	**-2***	1	0	-2
U_5	0	-8/3	-1	0	0	1	-1

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Dual Simplex Method

Iteration 1: Min{-2,-1} = -2; Min{-4/-1, -2/-1, -3/-2} = -3/-2

⇒ Pivot term -2*, U_3 -entering variable; U_3 -leaving variable.

	Z	U_1	U_2	U_3	U_4	U_5	RHS
Z	1	-5/2	-1/2	0	-3/2	0	3
U_3	0	1/2	1/2	1	-1/2	0	1
U_5	0	-8/3	**-1***	0	0	1	-1

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Dual Simplex Method

Iteration 2: Min{-(5/2)/-(8/3), -(1/2)/-1} = -(1/2)/-1

⇒ Pivot term -1*, U_2 -entering variable; U_5 -leaving variable.

	Z	U_1	U_2	U_3	U_4	U_5	RHS
Z	1	-7/6	0	0	-3/2	-1/2	7/2
U_3	0	-5/6	0	1	-1/2	1/2	1/2
U_2	0	8/3	1	0	0	-1	1

Optimal solution: $u_1^* = 0$, $u_2^* = -{}^1/_2$, $Z^* = -{}^7/_2$

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Important Parameters in the Simplex Tableau
Shadow Price (Dual Price)

Consider the product mix problem: Produce *n* products from *m* types of material.

- Available on-hand inventory of material i $(i = 1,2,\ldots,m)$: b_i
- Amount of material *i* used for one unit of product j $(j = 1,2,\ldots,n)$: a_{ij}
- Profit of one unit of product *j*: c_j'

Problem: determine production volumes of products so as to maximize total profit.

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Important Parameters in the Simplex Tableau
Shadow Price (Dual Price)

Denote x_j ($j = 1,2,..,n$): production volume of product *j*.

Objective Function: Maximize Profit $Z' = \sum_{j=1}^{n} c'_j x_j$

or Minimize "Cost" $Z = \sum_{j=1}^{n} c_j x_j$ with $c_j = -c'_j$

Constraints:

* Material constraints: $\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \forall i = 1,2,\ldots,m$

* Variable constraints: $x_j \geq 0 \quad \forall j = 1,2,\ldots,n$

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Important Parameters in the Simplex Tableau
Shadow Price (Dual Price)

It is noted that, slack variables $x_k (k = 1,2,\dots,m)$ will be introduced to convert the problem into standard form. The *optimal* simplex tableau of the standard form problem can be expressed as:

	z	x_j	x_k	RHS
z	1	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_j - c_j$	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{e}_k$	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_\mathbf{B}$	**0**	$\mathbf{B}^{-1}\mathbf{A}_j$	$\mathbf{B}^{-1}\mathbf{e}_k$	$\mathbf{B}^{-1}\mathbf{b}$

$\mathbf{e}_k$ is the unit column vector associated with x_k in the matrix **A**.

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Important Parameters in the Simplex Tableau
Shadow Price (Dual Price)

It should be noted that $\mathbf{c}_{\mathbf{B}}^{T}\mathbf{B}^{-1}\mathbf{e}_k = \left(\mathbf{c}_{\mathbf{B}}^{T}\mathbf{B}^{-1}\right)_k$

The optimal objective value: $Z^* = \mathbf{c}_{\mathbf{B}}^{T}\mathbf{B}^{-1}\mathbf{b} = \sum_{k=1}^{m}\left(\mathbf{c}_{\mathbf{B}}^{T}\mathbf{B}^{-1}\right)_k b_k$

Hence:

$$\frac{\partial z^*}{\partial b_k} = \left(\mathbf{c}_{\mathbf{B}}^{T}\mathbf{B}^{-1}\right)_k = \text{the } z\text{-value of } x_k \text{ at optimal solution}$$

$\left(\mathbf{c}_{\mathbf{B}}^{T}\mathbf{B}^{-1}\right)_k$ is called the shadow price of material k $(k = 1,2,\ldots,m)$. So, if the amount of material k increases (decreases) “one” unit, the optimal objective value will increase (decrease) $\left(\mathbf{c}_{\mathbf{B}}^{T}\mathbf{B}^{-1}\right)_k$ unit provided that the optimality condition is still satisfied.

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Important Parameters in the Simplex Tableau
Shadow Price (Dual Price)

Example 20: Produce two products A, B from three types of material 1,2, and 3.

1kg of product A requires: 1kg material 1, 1kg material 2 and 2kg material 3. Unit profit of A: 200 mil./ton

1kg of product B requires: 8/3kg material 1, 1kg material 2. Unit profit of B: 100 mil./ton

Available amount of materials 1,2, and 3: 4, 2, and 3 tons.

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Important Parameters in the Simplex Tableau
Shadow Price (Dual Price)

The LP program:

$$
\begin{aligned}
&\text{Min} \quad -2x_1 - x_2 \\
&\text{s.t} \\
&\quad x_1 + \frac{8}{3}x_2 \le 4 \\
&\quad x_1 + x_2 \le 2 \\
&\quad 2x_1 \le 3 \\
&\quad x_1, x_2 \ge 0
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
&\text{Min} \quad Z = -2x_1 - x_2 \\
&\text{s.t} \\
&\quad x_1 + \frac{8}{3}x_2 + x_3 = 4 \\
&\quad x_1 + x_2 + x_4 = 2 \\
&\quad 2x_1 + x_5 = 3 \\
&\quad x_1, x_2 \ge 0
\end{aligned}
$$

Monetary unit used in objective function: 100 millions.

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Important Parameters in the Simplex Tableau
Shadow Price (Dual Price)

The optimal simplex tableau:

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	0	0	0	-1	-1/2	-7/2
x_3	0	0	0	1	-8/3	5/6	7/6
x_2	0	0	1	0	1	-1/2	1/2
x_1	0	1	0	0	0	1/2	3/2

Optimal solution: $x_1^* = {}^{3}/_{2},\ x_2^* = {}^{1}/_{2},\ Z^* = -{}^{7}/_{2}$

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Important Parameters in the Simplex Tableau
Shadow Price (Dual Price)

From the optimal simplex tableau, it can be concluded that

1. Changing the amount of material 1 will not help increase the benefit
2. Increasing the amount of material 2 by 1 unit (1 ton) will help reduce the “cost” by 1 unit (100 millions), or equivalently, increase the profit by 1 unit (100 millions).
3. Increasing the amount of material 3 by 1 unit (1 ton) will help reduce the “cost” by 1/2 unit (50 millions), or equivalently, increase the profit by 1/2 unit (50 millions).

Note that the above analysis holds true only if the optimality condition is still satisfied when changing RHS parameters.

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Important Parameters in the Simplex Tableau
Reduced Cost

Consider the problem in example 20 and one of its simplex tableau during the solution process (not the optimal one!):

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	0	1	0	0	-1	-3
x_3	0	0	8/3	1	0	-1/2	5/2
x_4	0	0	1	0	1	-1/2	1/2
x_1	0	1	0	0	0	1/2	3/2

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Important Parameters in the Simplex Tableau
Reduced Cost

The value $c_j - \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{A}_j$ (which is the z-value with opposite sign) is called the reduced cost of producing product *j*.

For instances, in the above simplex tableau:

- Reduced costs associated with x_1, x_3, x_4 are 0.
- Reduced cost associated with x_2 is -1.
- Reduced cost associated with x_5 is 1.

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Important Parameters in the Simplex Tableau
Reduced Cost

Meaning of reduced cost:

Consider x_2, we have:

$$\mathbf{B}^{-1}\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 8/3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 1 \\ 0 \end{bmatrix}$$

This vector expresses the linear combination of basic variables which is equivalent to x_2. That is $x_2 \Leftrightarrow \frac{8}{3}x_3 + x_4 + 0x_1$.

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Important Parameters in the Simplex Tableau
Reduced Cost

In order to understand the equivalence; let assume that in addition to the production of the two products A, and B, we produce also C,D, and E with the following information:

1kg of product C requires: 1kg material 1. Unit profit of C: 0

1kg of product D requires: 1kg material 2. Unit profit of D: 0

1kg of product E requires: 1kg material 3. Unit profit of E: 0

The LP model with these new added products will not change and x_3, x_4, x_5 represent production volumes of C,D, and E, respectively.

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Important Parameters in the Simplex Tableau
Reduced Cost

The material consumptions of each unit of the products can be expressed by column vectors of the constraint matrix **A**:

$$\mathbf{A}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} 8/3 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{A}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{A}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{A}_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

It can be easily seen that the amount of each type of materials used to produce one kg of product B is exactly the same as the total amount of each type of materials used to produce 8/3kg of product C + 1kg of product D + 0kg of product A. The expression $x_2 \Leftrightarrow \frac{8}{3}x_3 + x_4 + 0x_1$ is used to illustrate the above relationship.

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Important Parameters in the Simplex Tableau
Reduced Cost

- $\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_2 = [0 \quad 0 \quad -2]\begin{bmatrix} 8/3 \\ 1 \\ 0 \end{bmatrix} = 0$: This value is the "cost" to produce the combination of (8/3kg of product C + 1kg of product D + 0kg of product A), which consume the same amount of materials as of 1kg of product B.

- Thus, $c_2 - \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_2 = -1$ is the reduction in total "cost" if 1kg of B is produced instead of the combination (8/3kg of product C + 1kg of product D + 0kg of product A).

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Important Parameters in the Simplex Tableau
Reduced Cost

So, if $c_2 - \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{A}_2 < 0$ (the z-value >0) as in this case, it will be better to produce B than its equivalent combination of (C,D,A) ⇒ x_2 should enter the basic solution to help reduce the objective value.

The above analysis of reduced cost also explains for the optimality condition in the primal simplex method discussed before.

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